Konuralp Journal of Mathematics, 10 (2) (2022) 240-249

Konuralp Journal of Mathematics

Research Paper

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath

e-ISSN: 2147-625X

Semi-Symmetric Generalized Sasakian Space Forms On Some Special Curvature Tensors

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Abstract

In this article, semi-symmetric generalized Sasakian space forms are investigated on some special curvature tensors. Characterizations of generalized Sasakian space forms are obtained on some specially selected $\sigma-$curvature tensors. By examining the flatness of these $\sigma-$curvature tensors, the properties of generalized sasakian space forms are given. More importantly, the cases of $\sigma-$semi-symmetric generalized Sasakian space forms are discussed and the behavior of the manifold is examined for each case. Again, necessary and sufficient conditions have been obtained for $\sigma-$symmetric generalized Sasakian space forms to be Einstein manifolds.

Keywords: *$\sigma-$Curvature Tensors, Semi-Symmetric Manifold, Sasakian Space Forms*
2010 Mathematics Subject Classification: *53C15; 53C44, 53D10*

1. Introduction

Let $M(\phi,\xi,\eta,g)$ be the almost contact metric manifold. If there are functions h_1,h_2,h_3 on M such that

$$
\begin{aligned}
& R(V_1,V_2)V_3 = h_1\left[g(V_2,V_3)V_1 - g(V_1,V_3)V_2\right] \\
& +h_2\left[g(V_1,\phi V_3)\phi V_2 - g(V_2,\phi V_3)\phi V_1\right. \\
& \left.+2g(V_1,\phi V_2)\phi V_3\right] + h_3\left[\eta(V_1)\eta(V_3)V_2\right. \\
& -\eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\xi \\
& \left.-g(V_2,V_3)\eta(V_1)\xi\right],
\end{aligned} \tag{1}
$$

$M=M(\phi,\xi,\eta,g)$ is called a generalized Sasakian space form and such a manifold is denoted by $M^{2n+1}(h_1,h_2,h_3)$. Such manifolds were introduced by P. Alegre et al [1]. P. Alegre, D. Blair and A. Carriazo calculated the Riemann curvature tensor of a generalized Sasakian space forms. In [2], generalized Sasakian space forms are studied under some conditions related to projective curvature. In this work, U.C. De and A. Sarkar obtained the necessary and sufficient conditions for generalized Sasakian space forms satisfying $PS=0$ and $PR=0$. Again, in [3], the same authors studied quasi conformal flat, Ricci symmetric and Ricci semi-symmetric generalized Sasakian space forms. In [4], the curvatures of para-Sasakian manifolds are studied and in this study C. Özgür and M.M. Tiripathi found necessary and sufficient conditions for the curvatures of para-Sasakian manifolds. M. Atçeken studied and classified generalized Sasakian space forms for some curvature conditions related to concircular, Riemann, Ricci and projective curvature tensors in [5]. Again, many authors have worked on generalized Sasakian space forms ([6]-[9]). In addition, M. Atçeken et al. have worked on many different manifolds ([11]-[17]).
In this article, semi-symmetric generalized Sasakian space forms are investigated on some special curvature tensors. Characterizations of generalized Sasakian space forms are obtained on some specially selected $\sigma-$curvature tensors. By examining the flatness of these $\sigma-$curvature tensors, the properties of generalized sasakian space forms are given. More importantly, the cases of $\sigma-$semi-symmetric generalized Sasakian space forms are discussed and the behavior of the manifold is examined for each case. Again, necessary and sufficient conditions have been obtained for $\sigma-$symmetric generalized Sasakian space forms to be Einstein manifolds.

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2. Preliminary

Let's take an $(2n+1)-$dimensional differentiable M manifold. If it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying the following conditions;

$\phi^2V_1 = -V_1 + \eta(V_1)\xi$ and $\eta(\xi) = 1,$

then this (ϕ,ξ,η) is called an almost contact structure, and the (M,ϕ,ξ,η) is called an almost contact manifold. If there is a g metric that satisfies the condition

$g(\phi V_1,\phi V_2) = g(V_1,V_2) - \eta(V_1)\eta(V_2)$ and $g(V_1,\xi) = \eta(V_1),$

for all $V_1,V_2 \in \chi(M)$ and $\xi \in \chi(M)$; (ϕ,ξ,η,g) is called almost contact metric structure and (M,ϕ,ξ,η,g) is called almost contact metric manifold. On the $(2n+1)$ dimensional M manifold,

$g(\phi V_1,V_2) = -g(V_1,\phi V_2)$

for all $V_1,V_2 \in \chi(M)$, that is, ϕ is an anti-symmetric tensor field according to the g metric. The Φ transformation defined as

$\Phi(V_1,V_2) = g(V_1,\phi V_2)$

for all $V_1,V_2 \in \chi(M)$, is called the fundamental 2-form of the (ϕ,ξ,η,g) almost contact metric structure, where

$\eta \wedge \Phi^n \neq 0.$

Sasakian space forms are very important for contact metric geometry. The curvature tensor for the Sasakian space form is defined as

$$R(V_1,V_2)V_3 = \left(\tfrac{c+3}{4}\right)[g(V_2,V_3)V_1 - g(V_1,V_3)V_2] + \left(\tfrac{c-1}{4}\right)[g(V_1,\phi V_3)\phi V_2 - g(V_2,\phi V_3)\phi V_1 + 2g(V_1,\phi V_2)\phi V_3 + \eta(V_1)\eta(V_3)V_2$$
$$-\eta(V_2)\eta(V_3)V_1 + g(V_1,V_3)\eta(V_2)\xi - g(V_2,V_3)\eta(V_1)\xi],$$

If we choose $h_1 = \frac{c+3}{4}, h_2 = h_3 = \frac{c-1}{4}$ in Sasakian space forms, then we get generalized Sasakian space forms.
If we choose $V_1 = \xi, V_2 = \xi$ and $V_3 = \xi$ respectively in (1), we obtain

$$R(\xi,V_2)V_3 = (h_1 - h_3)[g(V_2,V_3)\xi - \eta(V_3)V_2], \tag{2}$$

$$R(V_1,\xi)V_3 = (h_1 - h_3)[-g(V_1,V_3)\xi + \eta(V_3)V_2], \tag{3}$$

$$R(V_1,V_2)\xi = (h_1 - h_3)[\eta(V_2)V_1 - \eta(V_1)V_2]. \tag{4}$$

Also, if we take inner product of both sides of (1) by $\xi \in \chi(M)$, we get

$$\eta(R(V_1,V_2)V_3) = (h_1 - h_3)[g(V_2,V_3)\eta(V_1) - g(V_1,V_3)\eta(V_2)]. \tag{5}$$

Lemma 2.1. *For a $(2n+1)-$dimensional generalized Sasakian space form $M^{2n+1}(h_1,h_2,h_3)$ the following equations are provided [2].*

$$S(V_1,V_2) = [2nh_1 + 3h_2 - h_3]g(V_1,V_2) - [3h_2 + (2n-1)h_3]\eta(V_1)\eta(V_2), \tag{6}$$

$$S(V_1,\xi) = 2n(h_1 - h_3)\eta(V_1), \tag{7}$$

$$QV_1 = [2nh_1 + 3h_2 - h_3]V_1 - [3h_2 + (2n-1)h_3]\eta(V_1)\xi, \tag{8}$$

$$Q\xi = 2n(h_1 - h_3)\xi, \tag{9}$$

$$r = 2n(2n+1)h_1 + 6nh_2 - 4nh_3. \tag{10}$$

for all $V_1,V_2 \in \chi(M)$, where Q,S and r are the Ricci operator, Ricci tensor and scalar curvature of manifold $M^{2n+1}(h_1,h_2,h_3)$, respectively.

M. Tripathi and P. Gunam described a $T-$curvature tensor of the $(1,3)$ type in an n-dimensional (M,g) semi-Riemann manifold [10]. This curvature tensor is defined as

$$\begin{aligned} T(V_1,V_2)V_3 &= a_0R(V_1,V_2)V_3 + a_1S(V_2,V_3)V_1 + a_2S(V_1,V_3)V_2 + a_3S(V_1,V_2)V_3 + a_4g(V_2,V_3)QV_1 + a_5g(V_1,V_3)QV_2 \\ &+ a_6g(V_1,V_2)QV_3 + a_7r[g(V_2,V_3)V_1 - g(V_1,V_3)V_2], \end{aligned} \tag{11}$$

where R,S,Q, and r are Riemann curvature tensor, Ricci curvature tensor, Ricci operator, and scalar curvature of manifold M, respectively. According to the choosing of smooth functions $a_0,a_1,...,a_7$ the curvature tensor T is reduced to some special curvature tensors as follows.

Definition 2.2. *If $a_0 = 1, a_1 = -a_5 = -\frac{1}{2n}, a_2 = a_3 = a_4 = a_6 = a_7 = 0$ are chosen in (11), the σ_1-curvature tensor is defined as*

$$\sigma_1(V_1,V_2)V_3 = R(V_1,V_2)V_3 - \frac{1}{2n}[S(V_2,V_3)V_1 - g(V_1,V_3)QV_2]. \tag{12}$$

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (12), then we get

$$\sigma_1(\xi,V_2)V_3 = \frac{(1-2n)h_3 - 3h_2}{2n}[g(V_2,V_3)\xi - \eta(V_3)V_2], \tag{13}$$

$$\sigma_1(V_1,\xi)V_3 = 0, \tag{14}$$

$$\sigma_1(V_1,V_2)\xi = \frac{(1-2n)h_3 - 3h_2}{2n}[-\eta(V_1)V_2 + \eta(V_1)\eta(V_2)\xi]. \tag{15}$$

Definition 2.3. *If $a_0 = 1, a_1 = -a_5 = \frac{1}{2n}, a_2 = a_3 = a_4 = a_6 = a_7 = 0$ are chosen in (11), the σ_2-curvature tensor is defined as*

$$\sigma_2(V_1,V_2)V_3 = R(V_1,V_2)V_3 + \frac{1}{2n}[S(V_2,V_3)V_1 - g(V_1,V_3)QV_2]. \tag{16}$$

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (16), then we get

$$\sigma_2(\xi,V_2)V_3 = \frac{4nh_1 + 3h_2 - (2n+1)h_3}{2n}[g(V_2,V_3)\xi - \eta(V_3)V_2], \tag{17}$$

$$\sigma_2(V_1,\xi)V_3 = 2(h_1 - h_3)[-g(V_1,V_3)\xi + \eta(V_3)V_1], \tag{18}$$

$$\sigma_2(V_1,V_2)\xi = -\tfrac{4nh_2+3h_2-(2n+1)h_3}{2n}\eta(V_1)V_2 + 2(h_1-h_3)\eta(V_2)V_1 + \tfrac{3h_2+(2n-1)h_3}{2n}\eta(V_1)\eta(V_2)\xi. \tag{19}$$

Definition 2.4. *If $a_0 = 1, a_1 = -a_2 = \frac{1}{2n}, a_3 = a_4 = a_5 = a_6 = a_7 = 0$ are chosen in (11), the σ_3-curvature tensor is defined as*

$$\sigma_3(V_1,V_2)V_3 = R(V_1,V_2)V_3 + \frac{1}{2n}[S(V_2,V_3)V_1 - S(V_1,V_3)V_2]. \tag{20}$$

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (20), then we get

$$\sigma_3(\xi,V_2)V_3 = \tfrac{4nh_1+3h_2-(2n+1)h_3}{2n}g(V_2,V_3)\xi - 2(h_1-h_3)\eta(V_3)V_2 - \tfrac{3h_2+(2n-1)h_3}{2n}\eta(V_2)\eta(V_3)\xi, \tag{21}$$

$$\sigma_3(V_1,\xi)V_3 = -\tfrac{4nh_1+3h_2-(2n+1)h_3}{2n}g(V_1,V_3)\xi + 2(h_1-h_3)\eta(V_3)V_1 + \tfrac{3h_2+(2n-1)h_3}{2n}\eta(V_1)\eta(V_3)\xi, \tag{22}$$

$$\sigma(V_1,V_2)\xi = 2(h_1-h_3)[\eta(V_2)V_1 - \eta(V_1)V_2]. \tag{23}$$

Definition 2.5. *If $a_0 = 1, a_1 = -a_2 = -\frac{1}{2n}, a_3 = a_4 = a_5 = a_6 = a_7 = 0$ are chosen in (11), the σ_4-curvature tensor is defined as*

$$\sigma_4(V_1,V_2)V_3 = R(V_1,V_2)V_3 - \frac{1}{2n}[S(V_2,V_3)V_1 - S(V_1,V_3)V_2]. \tag{24}$$

For the $(2n+1)-$dimensional generalized Saskian space form, if we choose $V_1 = \xi, V_2 = \xi, V_3 = \xi$ respectively in (24), then we get

$$\sigma_4(\xi,V_2)V_3 = \frac{3h_2 + (2n-1)h_3}{2n}[-g(V_2,V_3)\xi + \eta(V_2)\eta(V_3)\xi], \tag{25}$$

$$\sigma_4(V_1,\xi)V_3 = \frac{3h_2 + (2n-1)h_3}{2n}[g(V_1,V_3)\xi - \eta(V_1)\eta(V_3)\xi], \tag{26}$$

$$\sigma_4(V_1,V_2)\xi = 0. \tag{27}$$

Definition 2.6. *Let M be a paracontact manifold. If its Ricci tensor S of type $(0,2)$ is of the form*

$$S(V_1,V_2) = ag(V_1,V_2) + b\eta(V_1)\eta(V_2),$$

then M is called $\eta-$Einstein manifold, where a,b are smooth functions on M. Also, if $b = 0$, then the manifold is called Einstein.

3. Flatness of $\sigma-$Curvature Tensors on Generalized Sasakian Space Forms

In this section, let's investigate the flatness of the $\sigma-$curvature tensors defined as above on generalized Sasakian space foms.

Theorem 3.1. *Let $M^{2n+1}(h_1,h_2,h_3)$ be a $(2n+1)-$dimensional gemeralized Sasakian space form. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_1-flat, then $M^{2n+1}(h_1,h_2,h_3)$ is an Einstein manifold provided $h_1 \neq h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_1-flat. So, we can write

$$\sigma_1(V_1,V_2)V_3 = 0$$

for every $V_1,V_2,V_3 \in \chi\left(M^{2n+1}\right)$. That is

$$R(V_1,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)V_1 - \frac{1}{2n}g(V_1,V_3)QV_2. \tag{28}$$

If we choose $V_1 = \xi$ in (28), we get

$$R(\xi,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)\xi - \frac{1}{2n}g(\xi,V_3)QV_2.$$

If we use (2) in the last equation, we have

$$(h_1-h_3)g(V_2,V_3)\xi - (h_1-h_3)\eta(V_3)V_2 = \tfrac{1}{2n}S(V_2,V_3)\xi - \tfrac{1}{2n}\eta(V_3)QV_2. \tag{29}$$

If we choose $V_3 = \xi$ in (29) and later, we take inner product both sides of (29) by $V_1 \in \chi\left(M^{2n+1}\right)$, we obtain

$$S(V_1,V_2) = 2n(h_1-h_3)g(V_1,V_2).$$

This completes the proof. □

Theorem 3.2. *Let $M^{2n+1}(h_1,h_2,h_3)$ be a $(2n+1)-$dimensional gemeralized Sasakian space form. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_2-flat, then $M^{2n+1}(h_1,h_2,h_3)$ is an $\eta-$Einstein manifold provided $h_1 \neq h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_2-flat. So, we can write

$$\sigma_2(V_1,V_2)V_3 = 0$$

for every $V_1,V_2,V_3 \in \chi\left(M^{2n+1}\right)$. That is

$$R(V_1,V_2)V_3 = -\frac{1}{2n}S(V_2,V_3)V_1 + \frac{1}{2n}g(V_1,V_3)QV_2. \tag{30}$$

If we choose $V_1 = \xi$ in (30), we get

$$R(\xi,V_2)V_3 = -\frac{1}{2n}S(V_2,V_3)\xi + \frac{1}{2n}g(\xi,V_3)QV_2.$$

If we use (2) in the last equation, we have

$$(h_1-h_3)[g(V_2,V_3)\xi - \eta(V_3)V_2] = -\frac{1}{2n}S(V_2,V_3)\xi + \frac{1}{2n}\eta(V_3)QV_2. \tag{31}$$

If we choose $V_3 = \xi$ in (31) and later, we take inner product both sides of (31) by $V_1 \in \chi\left(M^{2n+1}\right)$, we obtain

$$S(V_1,V_2) = 2n(h_1-h_3)[-g(V_1,V_2) + 2\eta(V_1)\eta(V_2)].$$

This completes the proof. □

Theorem 3.3. *Let $M^{2n+1}(h_1,h_2,h_3)$ be a $(2n+1)-$dimensional gemeralized Sasakian space form. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_3-flat, then $M^{2n+1}(h_1,h_2,h_3)$ is an $\eta-$Einstein manifold provided $h_1 \neq h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_3-flat. So, we can write

$$\sigma_3(V_1,V_2)V_3 = 0$$

for every $V_1,V_2,V_3 \in \chi(M)$. That is

$$R(V_1,V_2)V_3 = \frac{1}{2n}S(V_1,V_3)V_2 - \frac{1}{2n}S(V_2,V_3)V_1. \tag{32}$$

If we choose $V_1 = \xi$ in (32), we get

$$R(\xi,V_2)V_3 = \frac{1}{2n}S(\xi,V_3)V_2 - \frac{1}{2n}S(V_2,V_3)\xi.$$

If we use (2) in the last equation, we have

$$\begin{aligned} &(h_1-h_3)g(V_2,V_3)\xi - (h_1-h_3)\eta(V_3)V_2 \\ &= -\tfrac{1}{2n}S(V_2,V_3)\xi + (h_1-h_3)\eta(V_3)V_2. \end{aligned} \tag{33}$$

If we take inner product both sides of (33) by $\xi \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(V_2,V_3) = 2n(h_1-h_3)[-g(V_2,V_3) + 2\eta(V_2)\eta(V_3)].$$

This completes the proof. □

Theorem 3.4. *Let $M^{2n+1}(h_1,h_2,h_3)$ be a $(2n+1)-$dimensional gemeralized Sasakian space form. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_4-flat, then $M^{2n+1}(h_1,h_2,h_3)$ is an Einstein manifold provided $h_1 \neq h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_4-flat. So, we can write

$$\sigma_4(V_1,V_2)V_3 = 0$$

for every $V_1,V_2,V_3 \in \chi(M)$. That is

$$R(V_1,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)V_1 - \frac{1}{2n}S(V_1,V_3)V_2. \tag{34}$$

If we choose $V_1 = \xi$ in (34), we get

$$R(\xi,V_2)V_3 = \frac{1}{2n}S(V_2,V_3)\xi - \frac{1}{2n}S(\xi,V_3)V_2.$$

If we use (2) in the last equation, we have

$$\begin{aligned} &(h_1-h_3)g(V_2,V_3)\xi - (h_1-h_3)\eta(V_3)V_2 \\ &= \tfrac{1}{2n}S(V_2,V_3)\xi - (h_1-h_3)\eta(V_3)V_2. \end{aligned} \tag{35}$$

If we take inner product both sides of (35) by $\xi \in \chi(M)$ and make the necessary adjustments, we obtain

$$S(V_2,V_3) = 2n(h_1-h_3)g(V_2,V_3).$$

This completes the proof. □

4. Semi-Symmetric Generalized Sasakian Space Forms

In this section, the semi-symmetry condition of generalized Sasakian space forns will be investigated for some special σ-curvature tensors described above.

Theorem 4.1. *Let $M^{2n+1}(h_1,h_2,h_3)$ be the $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_1-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is either an $\eta-$Einstein manifold provided $h_3 \neq 2nh_1+3h_2$ and $3h_2 \neq (1-2n)h_3$ or $h_1 = h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_1-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)\sigma_1)(V_4,V_5,V_3) = 0,$$

for each $V_1,V_2,V_4,V_5,V_3 \in \chi\left(M^{2n+1}\right)$. That is, we can write

$$R(V_1,V_2)\sigma_1(V_4,V_5)V_3 - \sigma_1(R(V_1,V_2)V_4,V_5)V_3 - \sigma_1(V_4,R(V_1,V_2)V_5)V_3 - \sigma_1(V_4,V_5)R(V_1,V_2)V_3 = 0. \tag{36}$$

If we choose $V_1 = \xi$ in (36) and use (2), we get

$$\begin{aligned} &(h_1-h_3)\{g(V_2,\sigma_1(V_4,V_5)V_3)\xi - \eta(\sigma_1(V_4,V_5)V_3)V_2 \\ &-g(V_2,V_4)\sigma_1(\xi,V_5)V_3 + \eta(V_4)\sigma_1(V_2,V_5)V_3 \\ &-g(V_2,V_5)\sigma_1(V_4,\xi)V_3 + \eta(V_5)\sigma_1(V_4,V_2)V_3 \\ &-g(V_2,V_3)\sigma_1(V_4,V_5)\xi + \eta(V_3)\sigma_1(V_4,V_5)V_2\} = 0. \end{aligned} \tag{37}$$

If we use (13),(14),(15) in (37), we have

$$\begin{aligned} &(h_1-h_3)\{g(V_2,\sigma_1(V_4,V_5)V_3)\xi - \eta(\sigma_1(V_4,V_5)V_3)V_2 \\ &+\tfrac{(1-2n)h_3-3h_2}{2n}[-g(V_2,V_4)g(V_5,V_3)\xi + g(V_2,V_4)\eta(V_3)V_5 \\ &+g(V_2,V_3)\eta(V_4)V_5 - g(V_2,V_3)\eta(V_4)\eta(V_5)\xi] \\ &+\eta(V_4)\sigma_1(V_2,V_5)V_3 + \eta(V_5)\sigma_1(V_4,V_2)V_3 \\ &+\eta(V_3)\sigma_1(V_4,V_5)V_2\} = 0. \end{aligned} \tag{38}$$

If we choose $V_4 = \xi$ in (38) and make use of (13), we obtain

$$\begin{aligned} &(h_1-h_3)\Big\{-\tfrac{(1-2n)h_3-3h_2}{2n}g(V_5,V_3)V_2 + \sigma_1(V_2,V_5)V_3 \\ &+\tfrac{(1-2n)h_3-3h_2}{2n}g(V_2,V_3)V_5\Big\} = 0. \end{aligned} \tag{39}$$

Substituting (12) in (39), we have

$$(h_1-h_3)\left\{\frac{(1-2n)h_3-3h_2}{2n}\left[-g(V_5,V_3)V_2+g(V_2,V_3)V_5\right]\right.$$
$$\left.+R(V_2,V_5)V_3-\frac{1}{2n}S(V_5,V_3)V_2+\frac{1}{2n}g(V_2,V_3)QV_5\right\}=0. \tag{40}$$

If we choose $V_3=\xi$ in (40), we get

$$(h_1-h_3)\left\{-\frac{(1-2n)h_3-3h_2}{2n}\eta(V_5)V_2+\frac{1}{2n}\eta(V_2)QV_5\right.$$
$$\left.-\frac{2nh_1+3h_2-h_3}{2n}\eta(V_2)V_5\right\}=0. \tag{41}$$

If we choose $V_2=\xi$ in (41) and we take inner product both sides of the last equation by $V_3\in\chi\left(M^{2n+1}\right)$, we can write

$$(h_1-h_3)\left\{\frac{1}{2n}S(V_3,V_5)-\frac{2nh_1+3h_2-h_3}{2n}g(V_3,V_5)\right.$$
$$\left.-\frac{(1-2n)h_3-3h_2}{2n}\eta(V_3)\eta(V_5)\right\}=0.$$

From here, we have

$$S(V_3,V_5)=[2nh_1+3h_2-h_3]\,g(V_3,V_5)+[(1-2n)h_3-3h_2]\,\eta(V_3)\eta(V_5),$$

or

$$h_1=h_3.$$

This completes the proof. □

Theorem 4.2. *Let $M^{2n+1}(h_1,h_2,h_3)$ be $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_1-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is Einstein manifold if and only if $h_3\neq 2nh_1+3h_2$ and $3h_2=(1-2n)h_3$ relations must be provided*

Theorem 4.3. *Let $M^{2n+1}(h_1,h_2,h_3)$ be the $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_2-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is either an $\eta-$Einstein manifold provided $h_3\neq 2nh_1+3h_2$ and $3h_2\neq 2nh_3$ or $h_1=h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_2-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)\sigma_2)(V_4,V_5,V_3)=0,$$

for each $V_1,V_2,V_4,V_5,V_3\in\chi\left(M^{2n+1}\right)$. That is, we can write

$$R(V_1,V_2)\sigma_2(V_4,V_5)V_3-\sigma_2(R(V_1,V_2)V_4,V_5)V_3$$
$$-\sigma_2(V_4,R(V_1,V_2)V_5)V_3-\sigma_2(V_4,V_5)R(V_1,V_2)V_3=0. \tag{42}$$

If we choose $V_1=\xi$ in (42) and use (2), we get

$$(h_1-h_3)\{g(V_2,\sigma_2(V_4,V_5)V_3)\xi-\eta(\sigma_2(V_4,V_5)V_3)V_2$$
$$-g(V_2,V_4)\sigma_2(\xi,V_5)V_3+\eta(V_4)\sigma_2(V_2,V_5)V_3$$
$$-g(V_2,V_5)\sigma_2(V_4,\xi)V_3+\eta(V_5)\sigma_2(V_4,V_2)V_3$$
$$-g(V_2,V_3)\sigma_2(V_4,V_5)\xi+\eta(V_3)\sigma_2(V_4,V_5)V_2\}=0. \tag{43}$$

If we use (17),(18),(19) in (43), we have

$$(h_1-h_3)\{g(V_2,\sigma_2(V_4,V_5)V_3)\xi-\eta(\sigma_2(V_4,V_5)V_3)V_2$$
$$+\frac{4nh_1+3h_2-(2n+1)h_3}{2n}\left[-g(V_2,V_4)g(V_5,V_3)\xi+g(V_2,V_4)\eta(V_3)V_5\right.$$
$$\left.+g(V_2,V_3)\eta(V_4)V_5\right]+\eta(V_4)\sigma_2(V_2,V_5)V_3$$
$$+2(h_1-h_3)g(V_2,V_5)g(V_4,V_3)\xi$$
$$-2(h_1-h_3)g(V_2,V_5)\eta(V_3)V_4$$
$$+\eta(V_5)\sigma_2(V_4,V_2)V_3+\eta(V_3)\sigma_2(V_4,V_5)V_2$$
$$-2(h_1-h_3)g(V_2,V_3)\eta(V_5)V_4$$
$$\left.-\frac{3h_2+(2n-1)h_3}{2n}g(V_2,V_3)\eta(V_4)\eta(V_5)\xi\right\}=0. \tag{44}$$

If we choose $V_4 = \xi$ in (44) and make use of (17), we obtain

$$\begin{aligned}&(h_1-h_3)\left\{\tfrac{4nh_1+3h_2-(2n+1)h_3}{2n}\left[-g(V_5,V_3)V_2+g(V_2,V_3)V_5\right]\right.\\&\left.+\sigma_2(V_2,V_5)V_3+2h_3g(V_2,V_3)\eta(V_5)\xi\right\}=0.\end{aligned}\tag{45}$$

Substituting (16) in (45), we have

$$\begin{aligned}&(h_1-h_3)\left\{\tfrac{4nh_1+3h_2-(2n+1)h_3}{2n}\left[-g(V_5,V_3)V_2+g(V_2,V_3)V_5\right]\right.\\&+R(V_2,V_5)V_3+\tfrac{1}{2n}S(V_5,V_3)V_2-\tfrac{1}{2n}g(V_2,V_3)QV_5\\&\left.+2h_3g(V_2,V_3)\eta(V_5)\right\}\xi=0.\end{aligned}\tag{46}$$

If we choose $V_3 = \xi$ in (46), we get

$$\begin{aligned}&(h_1-h_3)\left\{-\tfrac{3h_2+2nh_3}{2n}\eta(V_5)V_2-\tfrac{1}{2n}\eta(V_2)QV_5\right.\\&\left.+\tfrac{2nh_1+3h_2-h_3}{2n}\eta(V_2)V_5+2h_3\eta(V_2)\eta(V_5)\xi\right\}=0.\end{aligned}\tag{47}$$

If we choose $V_2 = \xi$ in (47) and we take inner product both sides of the last equation by $V_3 \in \chi\left(M^{2n+1}\right)$, we can write

$$\begin{aligned}&(h_1-h_3)\left\{-\tfrac{1}{2n}S(V_3,V_5)+\tfrac{2nh_1+3h_2-h_3}{2n}g(V_3,V_5)\right.\\&\left.+\tfrac{-3h_2+2nh_3}{2n}\eta(V_3)\eta(V_5)\right\}=0.\end{aligned}$$

From here, we have

$$S(V_3,V_5)=[2nh_1+3h_2-h_3]\,g(V_3,V_5)+[2nh_3-3h_2]\,\eta(V_3)\eta(V_5),$$

or

$$h_1 = h_3.$$

This completes the proof. □

Theorem 4.4. *Let $M^{2n+1}(h_1,h_2,h_3)$ be $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_2-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is Einstein manifold if and only if $h_3 \neq 2nh_1+3h_2$ and $3h_2 = 2nh_3$ relations must be provided.*

Theorem 4.5. *Let $M^{2n+1}(h_1,h_2,h_3)$ be the $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_3-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is either an $\eta-$Einstein manifold provided $h_3 \neq 2nh_1+3h_2$ and $h_3 \neq h_1$ or $h_1 = h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_3-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)\sigma_3)(V_4,V_5,V_3)=0,$$

for each $V_1,V_2,V_4,V_5,V_3 \in \chi\left(M^{2n+1}\right)$. That is, we can write

$$\begin{aligned}&R(V_1,V_2)\sigma_3(V_4,V_5)V_3-\sigma_3(R(V_1,V_2)V_4,V_5)V_3\\&-\sigma_3(V_4,R(V_1,V_2)V_5)V_3-\sigma_3(V_4,V_5)R(V_1,V_2)V_3=0.\end{aligned}\tag{48}$$

If we choose $V_1 = \xi$ in (48) and use (2), we get

$$\begin{aligned}&(h_1-h_3)\{g(V_2,\sigma_3(V_4,V_5)V_3)\xi-\eta(\sigma_3(V_4,V_5)V_3)V_2\\&-g(V_2,V_4)\sigma_3(\xi,V_5)V_3+\eta(V_4)\sigma_3(V_2,V_5)V_3\\&-g(V_2,V_5)\sigma_3(V_4,\xi)V_3+\eta(V_5)\sigma_3(V_4,V_2)V_3\\&-g(V_2,V_3)\sigma_3(V_4,V_5)\xi+\eta(V_3)\sigma_3(V_4,V_5)V_2\}=0.\end{aligned}\tag{49}$$

If we use (21),(22),(23) in (49), we have

$$\begin{aligned}
&(h_1-h_3)\{g(V_2,\sigma_3(V_4,V_5)V_3)\xi-\eta(\sigma_3(V_4,V_5)V_3)V_2\\
&-\tfrac{4nh_1+3h_2-(2n+1)h_3}{2n}[g(V_2,V_4)g(V_5,V_3)\xi+g(V_2,V_5)g(V_4,V_3)\xi]\\
&+2(h_1-h_3)g(V_2,V_4)\eta(V_3)V_5+\eta(V_4)\sigma_3(V_2,V_5)V_3\\
&+\eta(V_5)\sigma_3(V_4,V_2)V_3+\eta(V_3)\sigma_3(V_4,V_5)V_2\\
&-\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_2,V_5)\eta(V_4)\eta(V_3)\xi-g(V_2,V_4)\eta(V_5)\eta(V_3)\xi]\\
&-2(h_1-h_3)[g(V_2,V_3)\eta(V_5)V_4-g(V_2,V_3)\eta(V_4)V_5\\
&+g(V_2,V_5)\eta(V_3)V_4]\}=0.
\end{aligned}\tag{50}$$

If we choose $V_4=\xi$ in (50) and make use of (21), we obtain

$$\begin{aligned}
&(h_1-h_3)\left\{\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_2,V_5)\eta(V_3)\xi+\eta(V_5)\eta(V_3)V_2\right.\\
&+g(V_2,V_3)\eta(V_5)\xi-\eta(V_5)\eta(V_2)\eta(V_3)\xi]+\sigma_3(V_2,V_5)V_3\\
&+2(h_1-h_3)g(V_2,V_3)V_5\}=0.
\end{aligned}\tag{51}$$

Substituting (20) in (51), we have

$$\begin{aligned}
&(h_1-h_3)\left\{\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_2,V_5)\eta(V_3)\xi+\eta(V_5)\eta(V_3)V_2\right.\\
&+g(V_2,V_3)\eta(V_5)\xi-2\eta(V_5)\eta(V_2)\eta(V_3)\xi]+R(V_2,V_5)V_3\\
&+\tfrac{1}{2n}S(V_5,V_3)V_2-\tfrac{1}{2n}S(V_2,V_3)V_5+2(h_1-h_3)g(V_2,V_3)V_5\}=0.
\end{aligned}\tag{52}$$

If we choose $V_5=\xi$ in (52) and we take inner product both sides of the last equation by $\xi\in\chi\left(M^{2n+1}\right)$, we can write

$$\begin{aligned}
&(h_1-h_3)\left\{-\tfrac{1}{2n}S(V_2,V_3)+\tfrac{2nh_1+3h_2-h_3}{2n}g(V_2,V_3)\right.\\
&+2(h_1-h_3)\eta(V_2)\eta(V_3)\}=0.
\end{aligned}$$

From here, we have

$$S(V_2,V_3)=[2nh_1+3h_2-h_3]g(V_2,V_3)+2(h_1-h_3)\eta(V_2)\eta(V_3),$$

or

$$h_1=h_3.$$

This completes the proof. □

Theorem 4.6. *Let $M^{2n+1}(h_1,h_2,h_3)$ be $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_3-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is Einstein manifold if and only if $h_3\neq 2nh_1+3h_2$ and $h_1=h_3$ relations must be provided.*

Theorem 4.7. *Let $M^{2n+1}(h_1,h_2,h_3)$ be the $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_4-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is either an $\eta-$Einstein manifold provided $h_3\neq 2nh_1+3h_2$ and $3h_2\neq(1-2n)h_3$ or $h_1=h_3$.*

Proof. Let's assume that $M^{2n+1}(h_1,h_2,h_3)$ is σ_4-semi symmetric manifold. Then we can write

$$(R(V_1,V_2)\sigma_4)(V_4,V_5,V_3)=0,$$

for each $V_1,V_2,V_4,V_5,V_3\in\chi\left(M^{2n+1}\right)$. That is, we can write

$$\begin{aligned}
&R(V_1,V_2)\sigma_4(V_4,V_5)V_3-\sigma_4(R(V_1,V_2)V_4,V_5)V_3\\
&-\sigma_4(V_4,R(V_1,V_2)V_5)V_3-\sigma_4(V_4,V_5)R(V_1,V_2)V_3=0.
\end{aligned}\tag{53}$$

If we choose $V_1=\xi$ in (53) and use (2), we get

$$\begin{aligned}
&(h_1-h_3)\{g(V_2,\sigma_4(V_4,V_5)V_3)\xi-\eta(\sigma_4(V_4,V_5)V_3)V_2\\
&-g(V_2,V_4)\sigma_4(\xi,V_5)V_3+\eta(V_4)\sigma_4(V_2,V_5)V_3\\
&-g(V_2,V_5)\sigma_4(V_4,\xi)V_3+\eta(V_5)\sigma_4(V_4,V_2)V_3\\
&-g(V_2,V_3)\sigma_4(V_4,V_5)\xi+\eta(V_3)\sigma_4(V_4,V_5)V_2\}=0.
\end{aligned}\tag{54}$$

If we use (25),(26),(27) in (54), we have

$$
\begin{aligned}
&(h_1-h_3)\{g(V_2,\sigma_4(V_4,V_5)V_3)\xi-\eta(\sigma_4(V_4,V_5)V_3)V_2\\
&+\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_2,V_4)g(V_5,V_3)\xi-g(V_2,V_4)\eta(V_5)\eta(V_3)\xi\\
&-g(V_2,V_5)g(V_4,V_3)\xi+g(V_2,V_5)\eta(V_4)\eta(V_3)\xi]\\
&+\eta(V_4)\sigma_4(V_2,V_5)V_3+\eta(V_5)\sigma_4(V_4,V_2)V_3\\
&+\eta(V_3)\sigma_4(V_4,V_5)V_2\}=0.
\end{aligned}
\tag{55}
$$

If we choose $V_4=\xi$ in (55) and make use of (25), we obtain

$$
\begin{aligned}
&(h_1-h_3)\left\{\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_5,V_3)V_2-\eta(V_5)\eta(V_3)V_2\right.\\
&-g(V_5,V_2)\eta(V_3)\xi-g(V_2,V_3)\eta(V_5)\xi+\eta(V_2)\eta(V_3)\eta(V_5)\xi]\\
&+\sigma_4(V_2,V_5)V_3\}=0.
\end{aligned}
\tag{56}
$$

Substituting (24) in (56), we have

$$
\begin{aligned}
&(h_1-h_3)\left\{\tfrac{3h_2+(2n-1)h_3}{2n}[g(V_5,V_3)V_2-\eta(V_5)\eta(V_3)V_2\right.\\
&-g(V_2,V_3)\eta(V_5)\xi-g(V_5,V_2)\eta(V_3)\xi+\eta(V_2)\eta(V_3)\eta(V_5)\xi]\\
&\left.+R(V_2,V_5)V_3-\tfrac{1}{2n}S(V_5,V_3)V_2+\tfrac{1}{2n}S(V_2,V_3)V_5\right\}=0.
\end{aligned}
\tag{57}
$$

If we choose $V_5=\xi$ in (57) and we take inner product both sides of the last equation by $\xi\in\chi\left(M^{2n+1}\right)$, we can write

$$
\begin{aligned}
&(h_1-h_3)\left\{\tfrac{1}{2n}S(V_2,V_3)-\tfrac{2nh_1+3h_2-h_3}{2n}g(V_2,V_3)\right.\\
&\left.-\tfrac{3h_2+(2n-1)h_3}{2n}\eta(V_2)\eta(V_3)\right\}=0.
\end{aligned}
$$

From here, we have

$$S(V_2,V_3)=[2nh_1+3h_2-h_3]g(V_2,V_3)+[3h_2+(2n-1)h_3]\eta(V_3)\eta(V_5),$$

or

$$h_1=h_3.$$

This completes the proof. □

Theorem 4.8. *Let $M^{2n+1}(h_1,h_2,h_3)$ be $(2n+1)-$dimensional generalized Sasakian space forms. If $M^{2n+1}(h_1,h_2,h_3)$ is σ_4-semi-symmetric, then $M^{2n+1}(h_1,h_2,h_3)$ is Einstein manifold if and only if $h_3\neq 2nh_1+3h_2$ and $3h_2=(1-2n)h_3$ relations must be provided.*

5. Conclusion

For many years, many studies have been done on the geometry of manifolds. This study has been prepared to contribute to making more detailed studies on generalized Sasakian Space Forms. In this article, semi-symmetric generalized Sasakian space forms are investigated on some special curvature tensors. Characterizations of generalized Sasakian space forms are obtained on some specially selected $\sigma-$curvature tensors. By examining the flatness of these $\sigma-$curvature tensors, the properties of generalized sasakian space forms are given. More importantly, the cases of $\sigma-$semi-symmetric generalized Sasakian space forms are discussed and the behavior of the manifold is examined for each case. Again, necessary and sufficient conditions have been obtained for $\sigma-$symmetric generalized Sasakian space forms to be Einstein manifolds.

Acknowledgements

The author would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

There are no competing interests.

Author's contributions

The author contributed to the writing of this paper. The author read and approved the final manuscript.

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